Notation : Dirac, Weyl and Majorana spinors

Let's start from Poincaré algebra (Lorentz + translation):

$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad (\det\Lambda = 1,\ \Lambda^0{}_0 \geq 1)$

generators: energy-momentum operator P^λ
angular momentum $M^{\mu\nu}$

$(P^\lambda)_\mu = i\delta^\lambda{}_\mu$
$(M^{\mu\nu})_{\rho\sigma} = i(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$

with algebra

(1) $[P^\lambda, P^\mu] = 0$

(2) $[M^{\mu\nu}, P^\lambda] = i(\eta^{\nu\lambda}P^\mu - \eta^{\mu\lambda}P^\nu)$

(3) $[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho})$

with metric $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

In susy the aim is to enlarge this by a spinor generator. We'll encounter Dirac, Weyl and Majorana spinors. To define these, we'll need γ-matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_4 \quad \text{(Clifford algebra)}$$

In Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \mu = 0,\dots,3$$

$$\sigma^\mu = (\mathbb{1}, \vec\sigma) \ , \quad \bar\sigma^\mu = (\mathbb{1}, -\vec\sigma) = \sigma_\mu$$

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

Dirac spinor

$$\psi_D = \psi_L + \psi_R \ , \quad \psi_L = \tfrac{1}{2}(\mathbb{1} - \gamma_5)\psi_D = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\psi_D$$

$$\psi_R = \tfrac{1}{2}(\mathbb{1} + \gamma_5)\psi_D = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\psi_D$$

with $\psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$ $\quad \alpha = 1,2$

$\psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot\alpha} \end{pmatrix}$ $\quad \dot\alpha = 1,2$

ψ_α, $\bar\chi^{\dot\alpha}$ are Weyl spinors. (often used in susy literature)

Dirac equation for charge $-e$ Dirac spinor

$$i\gamma^\mu(\partial_\mu - ieA_\mu)\psi_D = 0, \qquad \bar\psi = \psi^\dagger\gamma^0$$

$$-i\psi_D^\dagger\gamma^0(\partial_\mu + ieA_\mu)\gamma^0\gamma^{\mu\dagger} = 0 \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

$$-i\bar\psi_D(\partial_\mu + ieA_\mu)\gamma^\mu\gamma^0 = 0$$

$$\Rightarrow -i\gamma^{\mu T}(\partial_\mu + ieA_\mu)\bar\psi_D^T = 0$$

Find C such that $\boxed{-\gamma^{\mu T} = C^{-1}\gamma^\mu C}$

$$\Rightarrow i\gamma^\mu(\partial_\mu + ieA_\mu)\underbrace{C\bar\psi_D^T}_{\equiv\,\psi_D^c} = 0$$

$\equiv \psi_D^c$ with charge $+e$

Choose $C = -i\gamma^0\gamma^2$ in Weyl representation ; $C^T = C^\dagger = C^{-1} = -C$

$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$$

$$\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \Rightarrow \psi_D^c = C\gamma^0\psi_D^* = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_\alpha^* \\ \bar\chi^{\dot\alpha *} \end{pmatrix} = \begin{pmatrix} i\sigma^2\bar\chi^{\dot\alpha *} \\ -i\sigma^2\psi_\alpha^* \end{pmatrix}$$

Define $\bar\psi_{\dot\alpha} \equiv \psi_\alpha^*$, $\chi^\alpha = \bar\chi^{\dot\alpha *}$

$(i\sigma^2)_{\alpha\beta} \equiv \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ lowers undotted indices

$(-i\sigma^2)^{\dot\alpha\dot\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ raises dotted indices

$$\Rightarrow \psi_D^c = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}$$

It follows that

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta \quad , \quad \bar{\psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{\psi}^{\dot\beta}$$

with $\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = -i\sigma^2$

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = i\sigma^2$$

Majorana spinor is defined by

$$\psi_M^c = \psi_M$$

$$\Rightarrow \psi_\alpha = \chi_\alpha \quad (\text{or } \bar{\chi}^{\dot\alpha} = \bar{\psi}^{\dot\alpha})$$

Majorana spinor from a Weyl spinor :

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} \qquad (\bar{\psi}^{\dot\alpha} = (\psi^\alpha)^* = (\epsilon^{\alpha\beta}\psi_\beta)^*)$$

$$\psi_M^c = \underbrace{C}_{\begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}} \underbrace{\bar{\psi}_M^T}_{\gamma_0\psi_M^*} = \begin{pmatrix} i\sigma^2\bar{\psi}^{\dot\alpha *} \\ -i\sigma^2\psi_\alpha^* \end{pmatrix} = \begin{pmatrix} \epsilon_{\alpha\beta}\psi^\beta \\ \epsilon^{\dot\alpha\dot\beta}\bar{\psi}_{\dot\beta} \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} = \psi_M$$

Dirac spinor from two Majorana spinors :

$$\psi_D = \psi_{M_1} + i\psi_{M_2} \qquad \text{where} \quad \begin{cases} \psi_{M_1} = \frac{1}{2}(\psi_D + \psi_D^c) \\ \psi_{M_2} = \frac{1}{2i}(\psi_D - \psi_D^c) \end{cases}$$

$$\psi_D^c = C\gamma_0\psi_D^* \quad , \quad (\psi_D^c)^c = \psi_D$$

$$\Rightarrow \psi_{M_1}^c = \psi_{M_1} \quad , \quad \psi_{M_2}^c = \psi_{M_2}$$