

Session of Tuesday 14 February at 16-18 in aud A315

1. The propagator  $K(x, t) \equiv K(x, t; x = 0, t = 0)$  of a free particle in coordinate space is

$$K(x, t) = \theta(t) \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left[\frac{imx^2}{2\hbar t}\right]$$

- (a) Find the free propagator in momentum space,

$$\tilde{K}(p, t) \equiv \int_{-\infty}^{\infty} dx K(x, t) \exp(-ipx/\hbar)$$

- (b) Find the propagator as a function of momentum and energy,

$$\tilde{\tilde{K}}(p, E) \equiv \int_{-\infty}^{\infty} dt \tilde{K}(p, t) \exp[i(E + i\epsilon)t/\hbar]$$

and discuss the uncertainty relation: How the off-shellness  $E - p^2/2m$  in  $\tilde{\tilde{K}}(p, E)$  relates to the time interval  $t$  in  $\tilde{K}(p, t)$ .

2. Consider a particle subject to a one-dimensional harmonic oscillator potential  $V(x) = m\omega^2 x^2/2$ . Suppose at  $t = 0$  the state vector is given by  $|\ell\rangle = \exp(-i\hat{p}\ell/\hbar)|0\rangle$ , where  $\hat{p}$  is the momentum operator,  $\ell$  is some number with dimension of length and  $|0\rangle$  is the ground state of the oscillator.

- (a) Evaluate the expectation value  $\langle \ell | \hat{x}(t) | \ell \rangle$  for  $t \geq 0$ , where  $\hat{x}(t)$  is the position operator in the Heisenberg picture.  
 (b) Evaluate the correlation function

$$C(t) \equiv \langle 0 | \hat{x}(t) \hat{x}(0) | 0 \rangle$$

in the ground state  $|0\rangle$  of the harmonic oscillator.

You may use the expression for  $\hat{x}(t)$  that you worked out in problem 4(a) of Exercise 2.

3. (a) Write down the wave function (in coordinate space) at  $t = 0$  for the state  $|\ell\rangle$  specified in the previous problem. You may use

$$\langle x | 0 \rangle = \frac{1}{\pi^{1/4} \sqrt{x_0}} \exp\left[-\frac{1}{2} \left(\frac{x}{x_0}\right)^2\right], \quad x_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$$

- (b) Obtain an expression for the probability that the state is found in the ground state at  $t = 0$ . Does this probability change for  $t > 0$ ?

4. Derive the expression for the propagator  $\langle x | \exp(-iHt/\hbar) | x = 0 \rangle$  of a free particle from its path integral formulation,

$$\langle x, t | 0, 0 \rangle = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \hbar \Delta t}\right)^{\frac{1}{2}(N-1)} \int_{-\infty}^{\infty} dx_{N-1} \cdots dx_2 \exp\left[\frac{im}{2\hbar} \sum_{j=2}^N \Delta t \left(\frac{x_j - x_{j-1}}{\Delta t}\right)^2\right]$$

where  $x_1 = 0, x_N = x$  and  $t = (N - 1)\Delta t$ . Give also the expression for any finite  $N$ .

*Hint:* Use  $\xi_j = x_j - x_{j-1}$  ( $j = 2, \dots, N$ ) as integration variables by inserting a  $\delta$ -function constraint for  $x = \sum_{j=2}^N \xi_j$ .

Problems 2 and 3 are from J. J. Sakurai: *Modern Quantum Mechanics*, numbers 2.11+2.15 and 2.12, respectively.

$$1. K(x, t; 0, 0) = \theta(t) \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left[i \frac{m x^2}{2 \hbar t}\right]$$

$$\tilde{K}(p, t) \equiv \int_{-\infty}^{\infty} dx K(x, t; 0, 0) e^{-i p x / \hbar}$$

$$= \theta(t) \sqrt{\frac{m}{2\pi i \hbar t}} \int_{-\infty}^{\infty} dx \exp\left\{-\left(-i \frac{m}{2 \hbar t}\right) \left(x - \frac{p t}{m}\right)^2 - i \frac{p^2 t}{2 m \hbar}\right\}$$

$$= \theta(t) \exp\left(-i \frac{p^2}{2m} t / \hbar\right)$$

$$\tilde{K}(p, E) \equiv \int_{-\infty}^{\infty} dt K(p, t) e^{i(E + i\epsilon)t / \hbar}$$

$$= \int_0^{\infty} dt \exp\left[i\left(E - \frac{p^2}{2m} + i\epsilon\right)t / \hbar\right] =$$

$$= \frac{i}{E - \frac{p^2}{2m} + i\epsilon}$$

Relevant range of  $t$ -integral is  $t \lesssim \frac{1}{E - p^2/2m}$

Pole at  $E = \frac{p^2}{2m}$  arises from  $\infty$  range in  $t$

(not from size of integrand).

$$2 a) \quad x_H(t) = x_H(0) \cos \omega t + \frac{P_H(0)}{m\omega} \sin \omega t$$

$$\langle l | x_H(t) | l \rangle = \langle 0 | e^{iPl/\hbar} \left[ x \cos \omega t + \frac{P}{m\omega} \sin \omega t \right] e^{-iPl/\hbar} | 0 \rangle$$

$$\hookrightarrow \langle 0 | P | 0 \rangle = 0$$

$$[x, e^{-iPl/\hbar}] = [x, P] \frac{-i\ell}{\hbar} e^{-iPl/\hbar} = \ell e^{-iPl/\hbar}$$

$$\Rightarrow \langle l | x_H(t) | l \rangle = \ell \cos \omega t$$

$$b) \quad \langle 0 | x(t) x(0) | 0 \rangle = \langle 0 | x^2 \cos \omega t + \frac{Px}{m\omega} \sin \omega t | 0 \rangle$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \quad p = -i \sqrt{\frac{m\omega\hbar}{2}} (a - a^\dagger)$$

$$x^2 = \frac{\hbar}{2m\omega} [a^2 + \underset{\downarrow 1}{aa^\dagger} + a^\dagger a + (a^\dagger)^2] \quad [a, a^\dagger] = 1$$

$$px = -i \frac{\hbar}{2} (a - a^\dagger)(a + a^\dagger) = \frac{-i\hbar}{2} [a^2 + \underset{\downarrow 0}{aa^\dagger} + \underset{\leftarrow 1}{a^\dagger a} + \underset{\leftarrow 0}{(a^\dagger)^2}]$$

$$\langle 0 | x(t) x(0) | 0 \rangle = \frac{\hbar}{2m\omega} \cos \omega t - i \frac{\hbar}{2m\omega} \sin \omega t = \frac{\hbar}{2m\omega} e^{-i\omega t}$$

$$\begin{aligned}
 3. a) \langle x | l \rangle &= \langle x | e^{-i\hat{p}l/\hbar} | 0 \rangle = \int dp \langle x | p \rangle \langle p | e^{-i\hat{p}l/\hbar} | 0 \rangle \\
 &= \int \frac{dp}{2\pi\hbar} e^{-ipl/\hbar} e^{ipx/\hbar} \langle p | 0 \rangle \\
 &= \int \frac{dp}{2\pi\hbar} e^{ip(x-l)/\hbar} \langle p | 0 \rangle = \int \frac{dp}{2\pi\hbar} \langle x-l | p \rangle \langle p | 0 \rangle \\
 &= \langle x-l | 0 \rangle = \frac{1}{\pi^{1/4} \sqrt{x_0}} \exp\left[-\frac{1}{2} \frac{(x-l)^2}{x_0^2}\right], \quad x_0 = \sqrt{\frac{\hbar}{m\omega}}
 \end{aligned}$$

$$\begin{aligned}
 b) \langle 0 | l \rangle &= \int dx \langle 0 | x \rangle \langle x | l \rangle = \frac{1}{\sqrt{\pi} x_0} \int dx \exp\left[-\frac{x^2 + (x-l)^2}{2x_0^2}\right] \\
 &= \frac{1}{\sqrt{\pi} x_0} \int_{-\infty}^{\infty} dx \exp\left\{-\frac{1}{x_0^2} \left(x - \frac{1}{2}l\right)^2 + \frac{l^2 - 2l^2}{4x_0^2}\right\} = \exp\left[-\frac{l^2}{2x_0^2}\right]
 \end{aligned}$$

$$\text{Prob: } |\langle 0 | l \rangle|^2 = \exp\left[-\left(\frac{l}{x_0}\right)^2\right]$$

$$\langle 0 | e^{-iHt} | l \rangle = e^{-iE_0 t} \langle 0 | l \rangle \Rightarrow \text{Prob. is indep. of } t.$$

$$\langle x, t | 0, 0 \rangle = \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi i \hbar \Delta t} \right)^{(N-1)/2} \int_{-\infty}^{\infty} dx_{N-1} \dots dx_2$$

$$\times \exp \left[ \frac{i}{\hbar} \frac{1}{2} m \sum_{j=2}^N \frac{(x_j - x_{j-1})^2}{\Delta t} \right] \quad \begin{cases} x_N = x \\ x_1 = 0 \end{cases}$$

$$\text{Let } \xi_j = x_j - x_{j-1} \quad (j = 2, \dots, N)$$

$$\sum_{j=2}^N \xi_j = x_N - x_1 = x$$

$$\langle x, t | 0, 0 \rangle = \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{N-1}{2}} \int d\xi_2 \dots d\xi_N \frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{iux}$$

$$\times \prod_{j=2}^N \exp \left[ \frac{i}{\hbar} m \frac{\xi_j^2}{2\Delta t} - iu\xi_j \right]$$

where the  $u$ -integral gives  $\delta \left( \sum_{j=2}^N \xi_j - x \right)$ .

$$\sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_{-\infty}^{\infty} d\xi \exp \left\{ \frac{i}{\hbar} \frac{m}{2\Delta t} \left( \xi - u \frac{\hbar \Delta t}{m} \right)^2 - i \frac{\hbar \Delta t}{2m} u^2 \right\}$$

$$= \exp \left[ -i \frac{\hbar \Delta t}{2m} u^2 \right] \quad (N-1)\Delta t = t$$

$$\langle x, t | 0, 0 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp \left[ -i \frac{\hbar t}{2m} \left( u - \frac{m}{\hbar t} x \right)^2 + i \frac{m}{2\hbar t} x^2 \right]$$

$$= \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left[ i \frac{m}{2\hbar t} x^2 \right] \quad (\text{as on p. 30 of lecture notes})$$