Session of Tuesday 28 February at 16-18 in aud A315

1. Let $\mathcal{T}_{\mathbf{d}}$ be the translation operator (displacement vector $\mathbf{d}$); $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ the rotation operator, ($\hat{\mathbf{n}}$ and ϕ are the axis and angle of rotation, respectively) $\mathcal{P}$ the parity operator. Which, if any, of the following operator pairs commute, and why?
   (a) $\mathcal{T}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}'}$ ($\mathbf{d}$ and $\mathbf{d}'$ in different directions).
   (b) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ ja $\mathcal{D}(\hat{\mathbf{n}}', \phi')$ ($\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ in different directions).
   (c) $\mathcal{T}_{\mathbf{d}}$ and $\mathcal{P}$.
   (d) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and $\mathcal{P}$.

2. The particle state $|\ell s, jm\rangle$ of total angular momentum j, m can be constructed from the direct product of $|\ell, m_\ell\rangle$ (represented by the spherical harmonics $Y_\ell^m(\theta, \phi)$) and $|s, m_s\rangle$ (represented by two-component spinors for $s = \frac{1}{2}$).
   (a) Write the state $|0\frac{1}{2}, \frac{1}{2}\frac{1}{2}\rangle$ in the above representation.
   (b) Express $(\boldsymbol{\sigma} \cdot \boldsymbol{x})|0\frac{1}{2}, \frac{1}{2}\frac{1}{2}\rangle$ in terms of the $|\ell s, jm\rangle$ basis states.
   (c) Show that your result in (b) is understandable in view of the transformation properties of the operator $\boldsymbol{S} \cdot \boldsymbol{x}$ under rotations and under space inversion (parity).

3. (a) Evaluate: $\sum_{m=-j}^{j} m\, |d_{mm'}^{(j)}(\beta)|^2$ for *any* j; then check your answer for $j = \frac{1}{2}$.
   (b) Prove, for any j:

$$\sum_{m=-j}^{j} m^2\, |d_{m'm}^{(j)}(\beta)|^2 = \tfrac{1}{2} j(j+1) \sin^2\beta + m'^2 \tfrac{1}{2}(3\cos^2\beta - 1)$$

   [*Hint:* This can be proved in many ways. You may, for instance, examine the rotational properties of J_z^2 using the spherical (irreducible) tensor language.]

4. (a) Write xy, xz and $(x^2 - y^2)$ in terms of the components of the spherical (irreducible) tensor of spin (or rank) 2 formed from $T^{ij} = x^i x^j$, where $\boldsymbol{x} = (x, y, z)$.
   (b) The expectation value $Q \equiv e\langle j, m = j|(3z^2 - r^2)|j, m = j\rangle$ is known as the *quadrupole moment*. Evaluate

$$e\langle j, m'|(x^2 - y^2)|j, m = j\rangle$$

   (where $m' = j,\ j-1,\ j-2, \ldots$) in terms of Q and appropriate Clebsch-Gordan coefficients.

The problems are from J. J. Sakurai: *Modern Quantum Mechanics*, numbers 4.2, 4.4, 3.21 and 3.28, respectively.