31. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J M	J M	...
m_1 m_2	Coefficients		
m_1 m_2			
...			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2 × 1/2

m_1 m_2	1 +1
+1/2 +1/2	1

m_1 m_2	1 0	0 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	1 −1
−1/2 −1/2	1

1 × 1/2

m_1 m_2	3/2 +3/2
+1 +1/2	1

m_1 m_2	3/2 +1/2	1/2 +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2 −1/2	1/2 −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2 −3/2
−1 −1/2	1

2 × 1

m_1 m_2	3 +3
+2 +1	1

m_1 m_2	3 +2	2 +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3 +1	2 +1	1 +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	6/15	−1/2	1/10

m_1 m_2	3 0	2 0	1 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3 −1	2 −1	1 −1
0 −1	6/15	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3 −2	2 −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3 −3
−2 −1	1

1 × 1

m_1 m_2	2 +2
+1 +1	1

m_1 m_2	2 +1	1 +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2 0	1 0	0 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2 −1	1 −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2 −2
−1 −1	1

2 × 1/2

m_1 m_2	5/2 +5/2
+2 1/2	1

m_1 m_2	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2 −5/2
−2 −1/2	1

3/2 × 1/2

m_1 m_2	2 +2
+3/2 +1/2	1

m_1 m_2	2 +1	1 +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2 0	1 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2 −1	1 −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2 −2
−3/2 −1/2	1

3/2 × 1

m_1 m_2	5/2 +5/2
+3/2 +1	1

m_1 m_2	5/2 +3/2	3/2 +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2 −3/2	3/2 −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2 −5/2
−3/2 −1	1

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2 × 3/2

m_1 m_2	3 +3
+3/2 +3/2	1

m_1 m_2	3 +2	2 +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	3 +1	2 +1	1 +1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	3 0	2 0	1 0	0 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	3 −1	2 −1	1 −1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	3 −2	2 −2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	3 −3
−3/2 −3/2	1

2 × 3/2

m_1 m_2	7/2 +7/2
+2 +3/2	1

m_1 m_2	7/2 +5/2	5/2 +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	7/2 +3/2	5/2 +3/2	3/2 +3/2
+2 −1/2	1/7	16/35	2/5
+1 1/2	4/7	1/35	−2/5
0 3/2	2/7	−18/35	1/5

m_1 m_2	7/2 +1/2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 1/2	18/35	−3/35	−1/5	1/5
−1 3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	7/2 −1/2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 1/2	12/35	−5/14	0	3/10
−2 3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	7/2 −3/2	5/2 −3/2	3/2 −3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 1/2	1/7	−16/35	2/5

m_1 m_2	7/2 −5/2	5/2 −5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	7/2 −7/2
−2 −3/2	1

2 × 2

m_1 m_2	4 +4
+2 +2	1

m_1 m_2	4 +3	3 +3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	4 +2	3 +2	2 +2
+2 0	3/14	1/2	2/7
+1 1	4/7	0	−3/7
0 2	3/14	−1/2	2/7

m_1 m_2	4 +1	3 +1	2 +1	1 +1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 1	3/7	−1/5	−1/14	3/10
−1 2	1/14	−3/10	3/7	−1/5

m_1 m_2	4 0	3 0	2 0	1 0	0 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 1	8/35	−2/5	1/14	1/10	−1/5
−2 2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	4 −1	3 −1	2 −1	1 −1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 1	1/14	−3/10	3/7	−1/5

m_1 m_2	4 −2	3 −2	2 −2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	4 −3	3 −3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	4 −4
−2 −2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 31.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

Nabla operations in cylindrical (ρ, ϕ, z) coordinates

$$\nabla\psi = \mathbf{e}_\rho \frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\mathbf{A}_\rho) + \frac{1}{\rho}\frac{\partial\mathbf{A}_\phi}{\partial\phi} + \frac{\partial\mathbf{A}_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_\rho\left(\frac{1}{\rho}\frac{\partial\mathbf{A}_z}{\partial\phi} - \frac{\partial\mathbf{A}_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial\mathbf{A}_\rho}{\partial z} - \frac{\partial\mathbf{A}_z}{\partial\rho}\right) + \mathbf{e}_z\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho\mathbf{A}_\phi) - \frac{\partial\mathbf{A}_\rho}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Nabla operations in spherical (r, θ, ϕ) coordinates

$$\nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\mathbf{A}_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\mathbf{A}_\theta) + \frac{1}{r\sin\theta}\frac{\partial\mathbf{A}_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\mathbf{A}_\phi) - \frac{\partial\mathbf{A}_\theta}{\partial\phi}\right] + \mathbf{e}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial\mathbf{A}_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r\mathbf{A}_\phi)\right] + \mathbf{e}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(r\mathbf{A}_\theta) - \frac{\partial\mathbf{A}_r}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\text{Note that}\quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

Pauli and Dirac matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;; \qquad \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \;; \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$u(\boldsymbol{p}, s) = \frac{\not{p} + m}{\sqrt{E_p + m}}\begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} \;; \qquad v(\boldsymbol{p}, s) = \frac{-\not{p} + m}{\sqrt{E_p + m}}\begin{pmatrix} 0 \\ \varphi_{-s} \end{pmatrix}$$

Harmonic oscillator

$$V = \tfrac{1}{2}m\omega^2x^2, \quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2), \quad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$$

$$E_n = \hbar\omega(n + \tfrac{1}{2}), \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$\exp(\hat{A} + \hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-\lambda/2)$, when $[\hat{A}, \hat{B}] = \lambda$ is a c-number.

Spherical $\hat{T}_m^{j=1}$ vs. cartesian $\hat{\boldsymbol{x}}$ operator components: $T_0^1 = \hat{z}$, $T_{\pm1}^1 = \mp\frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$.