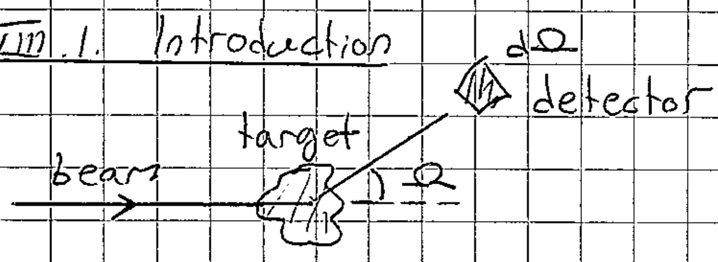
VIII Scattering theory

VIII.1. Introduction



- elastic scattering $A+B \rightarrow A+B$
  - same particles in the final state
- inelastic scattering: $A+B \Rightarrow C+D$ or $A+B \rightarrow \underset{\text{excited state}}{A^*} + B$

öbserve:
- particle species
- momenta (direction, magnitude)
- probability distribution $\rightarrow$ cross section
- spin / polarization

Cross section $\sigma = \dfrac{\text{number of scatterings per time}}{\text{incoming intensity}}$

differential cross section $d\sigma = \frac{d\sigma}{d\Omega} d\Omega$:
number of scatterings to angle $d\Omega$ per time / intensity

Intensity L = number of particles passing through a unit area per unit time

$$L = |\bar{J}|, \quad \bar{J} = -\frac{i\hbar}{2m}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right)$$

e.g. for plane wave $\psi = A e^{i\bar{p}\cdot\bar{r}/\hbar} \Rightarrow \bar{J} = |A|^2 \frac{\bar{p}}{m}$

$[\sigma] = m^2$, often used $1\ \text{barn} = 10^{-28}\ m^2 = 1\ b$

VIII.2. Scattering of plane wave

Scattering as time-independent problem, constant flow of particles. Later we study realistic wave packets.

Requirements for potential:

$\lim_{r\to\infty} r^3 V(\bar{r}) = 0$ asymptotically free enough

$\lim_{r\to 0} r^2 V(\bar{r}) = 0$ not too singular

Before scattering, momentum $\bar{p} = \hbar \bar{k}$, so outside the potential the incident wave is

$$\varphi_{\bar{k}}(\bar{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\bar{k}\cdot\bar{r}} \qquad \left(\text{normalized to } \delta^3(\bar{k}-\bar{k}')\right)$$

After scattering superposition of the original plane wave and outgoing spherical wave $\sim e^{ikr}/r$ (also a solution of free Schr. eq. with $\ell = 0$)

Inside the scattering region we have

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(\bar{r})\right]\psi_{\bar{k}}(\bar{r}) = E_k \psi_{\bar{k}}(\bar{r})$$

asymptotically kinetic en. $E_k = \frac{\hbar^2 k^2}{2\mu}$

μ - mass, if scattering off potential
- reduced mass, if two-particle scattering

Formally: $\hat{H}|\psi_{\bar{k}}\rangle = \left(\frac{\hat{\bar{p}}^2}{2\mu} + \hat{V}\right)|\psi_{\bar{k}}\rangle = E_k|\psi_{\bar{k}}\rangle$,

or, with $H_0 = \frac{p^2}{2\mu}$, $\quad (H_0 - E_k)|\psi_{\bar{k}}\rangle = -\hat{V}|\psi_{\bar{k}}\rangle$

Formal "solution"

$$|\psi_{\vec{k}}^{-}\rangle = |\varphi_{\vec{k}}\rangle - \frac{1}{H_0 - E_k}\hat{V}|\psi_{\vec{k}}^{-}\rangle$$

boundary condition; vanishes under $H_0 - E_k$

In coordinate repr.

$$\psi_{\vec{k}}^{-}(\vec{r}) = \langle\vec{r}|\psi_{\vec{k}}^{-}\rangle = \langle\vec{r}|\varphi_{\vec{k}}^{-}\rangle - \int d^3k'\, d^3k''\, \langle\vec{r}|\vec{k}'\rangle \times$$

$$\times \underbrace{\langle\vec{k}'|\frac{1}{H_0 - E_k}|\vec{k}''\rangle}_{\delta(\vec{k}'-\vec{k}'')\frac{1}{E_{k'}-E_k}}\langle\vec{k}''|\hat{V}|\psi_{\vec{k}}^{-}\rangle$$

$$= \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

$$- \int d^3k' \frac{e^{i\vec{k}'\cdot\vec{r}}}{(2\pi)^{3/2}} \frac{1}{\frac{\hbar^2 k'^2}{2\mu} - \frac{\hbar^2 k^2}{2\mu}} \int d^3r' d^3r'' \langle\vec{k}'|\vec{r}'\rangle \underbrace{\langle\vec{r}'|\hat{V}|\vec{r}''\rangle}_{\delta(\vec{r}'-\vec{r}'')V(\vec{r})}$$

$$\times \langle\vec{r}''|\psi_{\vec{k}}^{-}\rangle$$

$$= \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} - \frac{1}{(2\pi)^3}\frac{2\mu}{\hbar^2}\int d^3r'\, V(\vec{r}')\,\psi_{\vec{k}}^{-}(\vec{r}') \underbrace{\int d^3k' \frac{e^{i\vec{k}'\cdot(\vec{r}-\vec{r}')}}{k'^2 - k^2 \underset{(+)}{-} i\varepsilon}}_{\frac{2\pi^2}{|\vec{r}-\vec{r}'|} e^{\overset{+}{(-)} ik|\vec{r}-\vec{r}'|}}$$

$+i\varepsilon$ = incoming spherical wave

$$\Rightarrow \boxed{\psi_{\vec{k}}^{(+)}(\vec{r}) = \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} - \frac{2\mu}{4\pi\hbar^2}\int d^3r' \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\,\psi_{\vec{k}}^{(+)}(\vec{r}')}$$

Free particle propagator

Lippman-Schwinger equation

- integral equation, solved by iteration or matrix inversion numerically

Experimentally, detectors are far away from the scattering region

$\Rightarrow$ need asymptotic behavior, when $r \gg r'$

approximate $|\vec{r}-\vec{r}'| = \sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'} \doteq r\sqrt{1-\frac{2\vec{r}\cdot\vec{r}'}{r^2}+\left(\frac{r'}{r}\right)^2}$

$$\simeq r - \hat{r}\cdot\vec{r}' + \mathcal{O}\left(\frac{r'^2}{r^2}\right)$$

$\left\{\begin{array}{ll} r & \text{macroscopic} \\ r' & \text{microscopic} \end{array}\right.$

$\Rightarrow \lesssim 10^{-14}$

Asymptotically the scattered wave has $\vec{k}' \parallel \vec{r}$ and $|\vec{k}| = |\vec{k}'|$

$\Rightarrow$ denote $k\hat{r} = \vec{k}'$

For $r \gg r'$ we have then

$$\psi_{\vec{k}}^{(+)}(\vec{r}) \approx \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} - \frac{e^{ikr}}{4\pi r}\int d^3r'\, e^{-i\vec{k}'\cdot\vec{r}'}\, U(\vec{r}')\,\psi_{\vec{k}}^{(+)}(\vec{r}')$$

$\frac{2\mu}{\hbar^2}V(\vec{r}')$

$$\equiv \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} + \frac{F(\theta,\varphi)}{(2\pi)^{3/2}}\,\frac{e^{ikr}}{r}$$

scattering amplitude

outgoing spherical wave

$$\Rightarrow F(\theta,\varphi) = -\frac{(2\pi)^{3/2}}{4\pi}\int d^3r'\, e^{-i\vec{k}'\cdot\vec{r}'}\, U(\vec{r}')\,\psi_{\vec{k}}^{(+)}(\vec{r}')$$

Note: this looks like integral eq. for $\psi_{\vec{k}}^{(+)}$, but is valid only for large r, whereas small r wavefunction is needed inside the integral.

Current into solid angle $d\Omega$ is

$$\vec{j}\cdot d\vec{S} = -\frac{i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\cdot\hat{r}\, r^2 d\Omega$$

$$= -\frac{i\hbar r^2}{2m}\left(\psi^*\frac{\partial\psi}{\partial r} - \psi\frac{\partial\psi^*}{\partial r}\right)d\Omega = \frac{\hbar k}{m}\frac{|F(\theta,\varphi)|^2}{(2\pi)^3}\, d\Omega$$

Incoming intensity $\mathcal{I} = \frac{1}{(2\pi)^3}\frac{\hbar k}{m}$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = |F(\theta,\varphi)|^2}$$ differential cross section

Can also be written as

$$F(\theta,\varphi) = -2\pi^2\int d^3r'\,\varphi_{\vec{k}'}^*(\vec{r}')\,U(\vec{r}')\,\psi_{\vec{k}}^{(+)}(\vec{r}') = -2\pi^2\langle\vec{k}'|U|\psi_{\vec{k}}^{(+)}\rangle$$

VIII.3 Born approximation

$\hat{V}(\bar{r})$ distorts the initial plane wave $|\varphi_{\bar{k}}\rangle$ to $|\psi_{\bar{k}}^{(+)}\rangle$.
If V is small, approximate inside integral
$|\psi_{\bar{k}}^{(+)}\rangle \approx |\varphi_{\bar{k}}\rangle$ (cf. perturbation theory)

$\Rightarrow$ (first) Born approximation

$$F(\theta,\varphi) = -2\pi^2 \langle \bar{k}' | U | \bar{k} \rangle = -\frac{1}{4\pi} \int d^3r' \, e^{i(\bar{k}-\bar{k}')\cdot\bar{r}'} U(\bar{r}')$$

Fourier transform of potential.

If needed, higher terms from iteration

<u>Example:</u> Yukawa potential
(exhange of massive particle, mass m)

$$U(\bar{r}) = U_0 \frac{e^{-mr}}{r}$$

in Born approximation

$$F_{\bar{k}}(\bar{k}') = -\frac{U_0}{4\pi} \int d^3r' \frac{1}{r'} e^{-mr' + i(\bar{k}-\bar{k}')\cdot\bar{r}'}$$

$$= -\frac{U_0}{2i} \int_0^\infty dr' \, e^{-mr'} \left(e^{i|\bar{k}-\bar{k}'|r'} - e^{-i|\bar{k}-\bar{k}'|r'} \right) \frac{1}{|\bar{k}-\bar{k}'|}$$

$$= -\frac{U_0}{m^2 + \underbrace{(\bar{k}-\bar{k}')^2}_{\text{momentum transfer}}}$$

$$= -\frac{U_0}{m^2 + 4k^2 \sin^2 \theta/2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{U_0^2}{[m^2 + 4k^2 \sin^2 \theta/2]^2}$$

Let $m \to 0$, range $\to \infty$ $\Rightarrow$ Coulomb potential

$$U_0 = \frac{2\mu}{\hbar^2} \frac{e^2}{4\pi\varepsilon_0} \quad \rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{(4\pi\varepsilon_0)^2 \, 16 E^2 \sin^4 \theta/2}$$

$\Lsh$ Rutherford's classical result for Coulomb scattering. By accident, same result from class. theory

Singular at $\theta \to 0 \Rightarrow \sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \infty$

In reality, charge (nucleus) screened by other charges:

<u>More complicated example: electron scattering off atom</u>

$$H = H_0 + V; \quad H_0 = -\frac{\hbar^2}{2m}\nabla^2 + H_{atom}$$

$\Lsh$ electron

$$V = -\frac{Ze^2}{4\pi\varepsilon_0 r} + \sum_{j=1}^{Z} \frac{e^2}{4\pi\varepsilon_0 |\bar{r}-\bar{r}_j|}$$

$\Lsh$ j-th electron in atom

Elastic scattering $\to$ atom state does not change,

$$|a\bar{k}\rangle = |a\rangle \frac{1}{(2\pi)^{3/2}} e^{i\bar{k}\cdot\bar{r}} \longrightarrow |a\rangle \frac{1}{(2\pi)^{3/2}} e^{i\bar{k}'\cdot\bar{r}}$$

$\Lsh$ atom state

$$f_{\bar{k}}(\bar{k}') = -\frac{2m}{4\pi\hbar^2}\int d^3r\, e^{i(\bar{k}-\bar{k}')\cdot\bar{r}} \langle a|V|a\rangle$$

$$\langle a|V|a\rangle = -\frac{Ze^2}{4\pi\varepsilon_0 r} + V'_{aa}(\bar{r})$$

$\Lsh$ does not depend on atomic electrons

$$V'_{aa}(\bar{r}) = \langle a|\sum_{j=1}^{Z} \frac{e^2}{4\pi\varepsilon_0|\bar{r}-\bar{r}_j|}|a\rangle$$

$$= \int d^3r_1 \ldots d^3r_Z\, \varphi_a^*(\bar{r}_1, \ldots \bar{r}_Z)\varphi_a(\bar{r}_1, \ldots \bar{r}_Z)\sum_{j=1}^{Z}\frac{e^2}{4\pi\varepsilon_0|\bar{r}-\bar{r}_j|}$$

$$= \sum_{j=1}^{Z}\int d^3r_1 \ldots d^3r_Z\, \rho_a(\bar{r}_1, \ldots \bar{r}_Z)\frac{e^2}{4\pi\varepsilon_0|\bar{r}-\bar{r}_j|}$$

$$= Z\int d^3s\, \rho_a(\bar{s})\frac{e^2}{4\pi\varepsilon_0|\bar{r}-\bar{s}|}$$

$\Lsh$ prob. of finding an electron at $\bar{s}$, others anywhere. Z equal terms, as φ antisymm.

$$\to \langle a|V|a\rangle = -\frac{Ze^2}{4\pi\varepsilon_0}\left(\frac{1}{r} - \int d^3r \frac{\rho_a(\bar{s})}{|\bar{r}-\bar{s}|}\right)$$

Nucleus, Coulomb potential

Coulomb scattering from charge distribution

As earlier, $\int d^3r\, e^{i(\bar{k}-\bar{k}')\cdot\bar{r}} \frac{1}{r} = \frac{4\pi}{|\bar{k}-\bar{k}'|^2}$

Let $\bar{K} \equiv \bar{k} - \bar{k}'$:

$$\int d^3r\, e^{i\bar{K}\cdot\bar{r}} \int d^3s \frac{\rho_a(\bar{s})}{|\bar{r}-\bar{s}|} \underset{\bar{r}'=\bar{r}-\bar{s}}{=} \int d^3r' \frac{e^{i\bar{K}\cdot\bar{r}'}}{r'} \int d^3s\, e^{i\bar{K}\cdot\bar{s}} \rho_a(\bar{s})$$

$$= \frac{4\pi}{K^2} \tilde{\rho}_a(\bar{K}) \leftarrow \text{Fourier transform of } \rho_a$$

$$\Rightarrow \langle a\bar{k}'|V|a\bar{k}\rangle = -\frac{Ze^2}{\varepsilon_0 K^2}\left(1 - \tilde{\rho}_a(\bar{K})\right)$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2m \cdot Ze^2}{4\pi\varepsilon_0 \hbar^2 K^2}\right)^2 \left(1 - \tilde{\rho}_a(\bar{K})\right)^2$$

limits: $K \to 0 \Rightarrow \tilde{\rho}_a(k) \to \int d^3s\, \rho_a(s) = 1$ w.f. normalization

$K \to \infty \Rightarrow \tilde{\rho}_a(k) \to 0$ scattering from nucleus

Should have de Broglie wavelength << atom size to see structure. Too high K → only nucleus seen

VIII.4 Scattering of wave packets

Physically the beam consists of wave packets
⇒ time dependent problem

Gaussian minimum wave packet

$$\psi^{\bar{r}_0}_{\bar{k}_0}(\bar{r}) = \left(\frac{1}{a\sqrt{2\pi}}\right)^{3/2} e^{-\frac{(\bar{r}-\bar{r}_0)^2}{4a^2} + i\bar{k}_0\cdot(\bar{r}-\bar{r}_0)}$$

located around $\bar{r}_0$, $\langle\bar{r}\rangle = \bar{r}_0$, $\Delta x = \Delta y = \Delta z = a$
width a

In momentum space

$$\Phi^{\bar{r}_0}_{\bar{k}_0}(\bar{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3r\, \psi^{\bar{r}_0}_{\bar{k}_0}(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}} = \left(a\sqrt{\frac{2}{\pi}}\right)^{3/2} e^{-a^2(\bar{k}-\bar{k}_0)^2 - i\bar{k}\cdot\bar{r}_0}$$

$\langle\bar{k}\rangle = \bar{k}_0$, $\Delta k_x = \frac{1}{2a}$

Initially, particle (wave packet) at $\bar{r}_0 = (0, 0, -z_0)$
$z_0 \gg R$, R range of potential, and beam directed along z-axis, $\bar{k}_0 = (0, 0, k_0)$

$$\Rightarrow \psi(\bar{r}, t=0) = \Psi^{\bar{r}_0}_{\bar{k}_0}(\bar{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, \Phi^{\bar{r}_0}_{\bar{k}_0}(\bar{k}) \, e^{i\bar{k}\cdot\bar{r}}$$

For energy eigenstates time development known.
When $V \neq 0$, $\psi^{(+)}_{\bar{k}}(\bar{r})$, not $e^{i\bar{k}\cdot\bar{r}}$, is eigenstate of H.

From Lippmann - Schwinger eq.

$$\frac{1}{(2\pi)^{3/2}} e^{i\bar{k}\cdot\bar{r}} = \psi^{(+)}_{\bar{k}}(\bar{r}) + \frac{1}{4\pi} \int d^3r' \, \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} U(\bar{r}') \psi^{(+)}_{\bar{k}}(\bar{r}')$$

Substituted above, the last term gives exponentially small contribution (see CM 7.6): at $t=0$ w.f exponentially small where $V \neq 0$.

$$\Rightarrow \quad \psi(\bar{r}, 0) = \Psi^{\bar{r}_0}_{\bar{k}_0}(\bar{r}) \approx \int d^3k \, \Phi^{\bar{r}_0}_{\bar{k}_0}(\bar{k}) \, \psi^{(+)}_{\bar{k}}(\bar{r})$$

Time dependence

$$\psi(\bar{r}, t) = \int d^3k \, \Phi^{\bar{r}_0}_{\bar{k}_0}(\bar{k}) \, \psi^{(+)}_{\bar{k}}(\bar{r}) \, e^{-i\frac{\hbar k^2}{2\mu}t}$$

$$\underset{\text{large } r}{\approx} \frac{1}{(2\pi)^{3/2}} \int d^3k \, \Phi^{\bar{r}_0}_{\bar{k}_0}(\bar{k}) \left[e^{i\bar{k}\cdot\bar{r}} + F_{\bar{k}}(\theta, \varphi) \frac{e^{ikr}}{r} \right] e^{-\frac{i\hbar k^2 t}{2\mu}}$$

$\Phi^{\bar{r}_0}_{\bar{k}_0}(\bar{k})$ vanishes, unless $\bar{k} \approx \bar{k}_0$, approximate $k^2 = (\bar{k}-\bar{k}_0)^2 - k_0^2 + 2\bar{k}\cdot\bar{k}_0$
$\approx -k_0^2 + 2\bar{k}\cdot\bar{k}_0$

$$\Rightarrow \psi(\bar{r}, t) \approx \underbrace{e^{\frac{i\hbar k_0^2 t}{2\mu}}}_{\text{common phase, not important}} \frac{1}{(2\pi)^{3/2}} \int d^3k \, \Phi^{\bar{r}_0}_{\bar{k}_0}(\bar{k}) \left[e^{i\bar{k}\cdot(\bar{r} - \frac{\hbar \bar{k}_0}{\mu}t)} + \underbrace{\cancel{\frac{F(\theta,\varphi)}{r}}}_{\psi_{scatt}} \right]$$

$$\psi_{scatt} = F_{\bar{k}}(\theta, \varphi) \frac{e^{ikr}}{r} e^{-i\bar{k}\cdot\frac{\hbar \bar{k}_0}{\mu}t}$$

First term : $\frac{1}{(2\pi)^{3/2}} \int d^3k \, \Phi_{\bar{k}_0}^{\bar{r}_0}(\bar{k}) \, e^{i\bar{k}\cdot(\bar{r} - \frac{\hbar \bar{k}_0}{\mu} t)} = \psi_{\bar{k}_0}^{\bar{r}_0}(\bar{r} - \bar{v}_0 t)$

$\bar{v}_0 = \frac{\hbar \bar{k}_0}{\mu} = \frac{\bar{p}_0}{\mu}$ translated wave packet,

nonzero when $\bar{r} - \bar{v}_0 t \approx \bar{r}_0 \Leftrightarrow \bar{r} \approx \bar{r}_0 + \bar{v}_0 t$

Second term (scattered wave); approximate $\bar{k} \approx \bar{k}_0$

$$\frac{1}{(2\pi)^{3/2}} \int d^3k \, \Phi_{\bar{k}_0}^{\bar{r}_0}(\bar{k}) \, F_{\bar{k}}(\theta,\varphi) \frac{e^{ikr}}{r} e^{-i\bar{k}\cdot \frac{\hbar \bar{k}_0 t}{\mu}}$$

$$\approx \frac{F_{\bar{k}_0}(\theta,\varphi)}{r} \frac{1}{(2\pi)^{3/2}} \int d^3k \, \Phi_{\bar{k}_0}^{\bar{r}_0}(\bar{k}) e^{i\bar{k}\cdot(\hat{k}_0 r - \bar{v}_0 t)}$$

$$= \frac{1}{r} F_{\bar{k}_0}(\theta,\varphi) \, \psi_{\bar{k}_0}^{\bar{r}_0}(\hat{k}_0 r - \bar{v}_0 t)$$

Nonzero when $\hat{k}_0 r - \bar{v}_0 t \approx \bar{r}_0 \Leftrightarrow r - v_0 t = -z_0$
all in z-direction

$\Leftrightarrow \quad r = -z_0 + v_0 t$

spherical surface expanding with time (after collision at $t = z_0/v_0$)

From the wave packet definition, current through $d\Omega$ is

$$\bar{F}\cdot d\bar{S} = \frac{-i\hbar}{2\mu}(\psi^* \nabla\psi - \psi\nabla\psi^*)\cdot \hat{r}\, r^2 d\Omega = v_0 |F_{\bar{k}_0}(\hat{r})|^2 |\psi_{\bar{k}_0}^{\bar{r}_0}(r\hat{k}_0 - \bar{v}_0 t)|^2 d\Omega$$

integrated over all times:

$$P(d\Omega) = v_0 |F_{\bar{k}_0}(\hat{r})|^2 d\Omega \int_{-\infty}^{\infty} dt \, |\psi_{\bar{k}_0}^{\bar{r}_0}(r\hat{k}_0 - \bar{v}_0 t)|^2$$

$$= |F_{\bar{k}_0}(\hat{r})|^2 d\Omega \int_{-\infty}^{\infty} dz \, |\psi_{\bar{k}_0}^{\bar{r}_0}(0,0,z)|^2$$

Incident intensity is

$$\bar{F}\cdot d\bar{S} = v_0 |\psi_{\bar{k}_0}^{\bar{r}_0}(\bar{r} - \bar{v}_0 t)|^2 dS \xrightarrow{\int dt} \int_{-\infty}^{\infty} dz \, |\psi(0,0,z)|^2$$

$\leftarrow dS \perp \hat{z}$, dS on z-axis

$$\Rightarrow \frac{d\sigma}{d\Omega} = |F_{\bar{k}_0}(\hat{r})|^2$$

VIII.5 Partial wave analysis

Spherically symmetric potential, choose z-axis in beam direction $\Rightarrow$ scattering amplitude $F(\Theta,\varphi)$ independent of φ.

Angular dependence in terms of spherical harmonics

$$\psi^{(+)}_{\vec{k}}(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_{lm}(k,r)\, Y_{lm}(\Theta,\varphi) \qquad \Bigg| \; Y_{l0} = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\Theta)$$

$$\underset{\text{only } m=0}{=} \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}}\, R_{l0}(k,r)\, P_l(\cos\Theta)$$

R_l satisfies the radial equation

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} - \underbrace{\frac{2\mu}{\hbar^2} V(r)}_{U(r)}\right] R_l(k,r) = 0$$

and the asymptotic condition is

$$\psi^{(+)}_{\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\left(e^{i\vec{k}\cdot\vec{r}} + f_{\vec{k}}(\Theta)\frac{e^{ikr}}{r}\right)$$

Assumed that $V(r)$ vanishes faster than $\frac{1}{r^3}$

$\rightarrow$ asymptotically free equation, solutions are spherical Bessel functions (cf. lec notes I.3. b))

$$R_l(k,r) \approx A_l(k)\, j_l(kr) + B_l(k)\, n_l(kr)$$

Large distance limit of this is

$$\begin{cases} j_l(kr) \approx \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right)\cdot\left(1 + O\left(\frac{1}{kr}\right)\right) \\ n_l(kr) = -\frac{1}{kr}\cos\left(kr - \frac{l\pi}{2}\right)\left(1 + O\left(\frac{1}{kr}\right)\right) \end{cases}$$

Define $C_l(k)$ & $\delta_l(k)$: $\begin{cases} A_l(k) = C_l(k)\cos\delta_l(k) \\ B_l(k) = -C_l(k)\sin\delta_l(k) \end{cases}$

$$\Rightarrow \quad R_l(k,r) \approx \frac{C_l(k)}{kr}\left[\cos\delta_l(k)\sin\left(kr - \frac{l\pi}{2}\right) + \sin\delta_l(k)\cos\left(kr - \frac{l\pi}{2}\right)\right]$$

$$= \frac{C_l(k)}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l(k)\right)$$

Argument shifted w.r.t. free solution
$\Rightarrow$ phase shift $\delta_\ell(k)$ $\delta > 0$ attraction
$\delta < 0$ repulsion

To compare with the known asymptotic form, need plane wave in spherical coordinates:

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^\ell\, j_\ell(kr)\, P_\ell(\underbrace{\hat{k}\cdot\hat{r}}_{\cos\theta})$$

$$\Rightarrow \psi_{\vec{k}}^{(+)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1) P_\ell(\cos\theta)\left[i^\ell j_\ell(kr) + f_\ell(k)\frac{e^{ikr}}{r}\right]$$

$$\approx \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1) P_\ell(\cos\theta)\left[\frac{i^\ell}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right) + f_\ell\frac{e^{ikr}}{r}\right]$$

$f_{\vec{k}}(\theta) \equiv \sum_\ell (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$

P_ℓ independent $\Rightarrow$ coefficients agree:

$$\sqrt{\frac{2\ell+1}{4\pi}}\, C_\ell(k)\, \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} = \frac{2\ell+1}{(2\pi)^{3/2}}\left[i^\ell \frac{\sin(kr - \frac{\ell\pi}{2})}{kr} + f_\ell \frac{e^{ikr}}{r}\right]$$

$$\Leftrightarrow C_\ell(k)\left[e^{i(kr - \frac{\ell\pi}{2} + \delta_\ell)} - e^{-i(kr - \frac{\ell\pi}{2} + \delta_\ell)}\right]$$
$$= \sqrt{\frac{2\ell+1}{2}}\,\frac{1}{\pi}\left[i^\ell\left(e^{i(kr - \frac{\ell\pi}{2})} - e^{-i(kr - \frac{\ell\pi}{2})}\right) + 2ikf_\ell\, e^{ikr}\right]$$

e^{ikr} and e^{-ikr} linearly independent, $e^{\pm i\frac{\ell\pi}{2}} = (\pm i)^\ell$

$$\Rightarrow \begin{cases} C_\ell(-i)^\ell e^{i\delta_\ell} = \sqrt{\frac{2\ell+1}{2\pi^2}}\,(1 + 2ikf_\ell) \\ -C_\ell\, i^\ell\, e^{-i\delta_\ell} = -\sqrt{\frac{2\ell+1}{2\pi^2}}\,(-1)^\ell \end{cases}$$

$$\Rightarrow e^{2i\delta_\ell} = 1 + 2ikf_\ell, \quad \boxed{f_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k}}$$

$$C_\ell = i^\ell e^{i\delta_\ell}\sqrt{\frac{2\ell+1}{2\pi^2}}$$

Partial wave scattering amplitude

Scattering amplitude

$$f_{\vec{k}}(\theta) = \sum_\ell (2\ell+1)\frac{e^{i\delta_\ell}\sin\delta_\ell}{k} P_\ell(\cos\theta)$$

Total ~~scattering amplitude~~ cross section

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega = \int d\Omega \, |f_k(\theta)|^2 =$$

$$= 2\pi \int_{-1}^{1} d(\cos\theta) \sum_{\ell\ell'} (2\ell+1)(2\ell'+1) \frac{e^{i(\delta_\ell - \delta_{\ell'})} \sin\delta_\ell \sin\delta_{\ell'}}{k^2} \underbrace{P_\ell(\cos\theta) P_{\ell'}(\cos\theta)}_{\text{orthog.: } \frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$= \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2\delta_\ell$$

On the other hand, the forward amplitude is $(P_\ell(1) = 1)$

$$f(\theta = 0) = \sum_\ell (2\ell+1) \frac{e^{i\delta_\ell} \sin\delta_\ell}{k} = \sum_\ell (2\ell+1) \frac{\cos\delta_\ell \sin\delta_\ell + i \sin^2\delta_\ell}{k}$$

$$\Rightarrow \boxed{\sigma_{tot} = \frac{4\pi}{k} \operatorname{Im} f_k(\theta = 0)} \quad \text{Optical theorem}$$

Valid also when inelastic processes present.
Related to unitarity ≈ probability conservation

δ_ℓ computed from radial Schrödinger eq.

Example: S-wave scattering from spherical potential wall ("soft sphere")

$$V(\bar{r}) = \begin{cases} V_0 > 0, & r < a \\ 0, & r > a \end{cases}$$

$\ell = 0$ eq.

$$\begin{array}{lll} u''(r) + k^2 u(r) = 0 & r > a & \\ u''(r) + K^2 u(r) = 0 & r < a, \; E > V_0 & K = \sqrt{\frac{2m}{\hbar^2}(E - V_0)} \\ \text{or} \quad u''(r) - \varkappa^2 u(r) = 0 & r < a, \; E < V_0 & \varkappa = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \end{array}$$

Regular solution $(u(0) = 0)$ inside potential:
$u \sim \sin Kr$ or $\sinh \varkappa r$

$$\Rightarrow R_0(r) = A \frac{\sin Kr}{Kr} = A\, j_0(Kr) \quad \text{or} \quad R_0 = A \frac{\sinh \varkappa r}{\varkappa r}$$

Outside potential $R_0(r) = B \dfrac{\sin(kr+\delta_0)}{kr} e^{i\delta_0}$
$\left(B = \frac{1}{\sqrt{2\pi^2}}\right)$

$E > V_0$

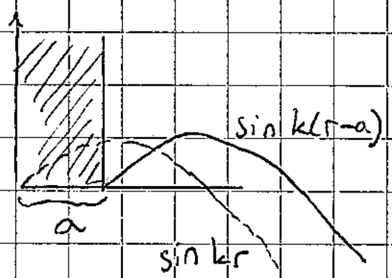
Continuity at $r=a \Rightarrow \begin{cases} A \sin Ka = B e^{i\delta_0} \sin(ka+\delta_0) \\ AK \cos Ka = kB e^{i\delta_0} \cos(ka+\delta_0) \end{cases}$

$$\Rightarrow \frac{\tan Ka}{K} = \frac{\tan(ka+\delta_0)}{k} \Rightarrow \tan\delta_0 = \frac{k\tan Ka - K\tan ka}{K + k\tan Ka \tan ka}$$

$E < V_0$;
$$\tan\delta_0 = \frac{k\tanh \varkappa a - \varkappa \tan ka}{\varkappa + k\tan\varkappa a \tan ka}$$

Hard sphere limit: $V_0 \to \infty \Rightarrow \varkappa \to \infty$, $\tanh \varkappa a \to 1$

$$\Rightarrow \tan\delta_0 = -\tan ka \Rightarrow \delta_0 \to -ka$$



$$\sigma = \frac{4\pi}{k^2}\sin^2 ka \xrightarrow[k\to 0]{} 4\pi a^2,$$

whole sphere at low energy, 4 times classical value

softer sphere at low energy: $(k \approx 0)$

$$\tan\delta_0 \approx k\left(\frac{1}{\varkappa_0}\tanh \varkappa_0 a - a\right) \approx \delta_0 \quad , \varkappa_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$$

$$\Rightarrow \delta_0 = k \cdot \underbrace{\text{const}}_{-\text{ scattering length}}$$