KVANTTIMEKANIIKKA I Autumn 2011 Exercise 9

Due at 8.15 on Thursday 17 November 2011

1. Considering the commutation of translation and rotation generators ($\hat{p}_i$ and $\hat{L}_i$) with the Hamiltonian operator, what are the constants of motion in the following three-dimensional potentials:
   (a) $V(\mathbf{r}) = f(\sqrt{x^2 + y^2})$
   (b) $V(\mathbf{r}) = \lambda z$
   (c) $V(\mathbf{r}) = \lambda z f(r)$
2. Show that the operator $U_{\mathbf{a}} = e^{-i\mathbf{a}\cdot\mathbf{p}/\hbar}$ is a translation operator for functions, $\mathbf{r} \to \mathbf{r} + \mathbf{a}$. Here $\mathbf{a}$ is a constant vector and $\mathbf{p}$ the momentum *operator*. (Hint: think of Taylor series).
3. Express $\hat{J}_x$ and $\hat{J}_y$ in terms of $\hat{J}_+$ and $\hat{J}_-$ and hence show that for a system in an eigenstate of $\hat{J}_z$, $\langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0$. Also obtain expressions for $\langle \hat{J}_x^2 \rangle$ and $\langle \hat{J}_y^2 \rangle$ and compare the product of these two quantities with the predictions of the general uncertainty principle, $(\Delta A)^2 (\Delta B)^2 \geq |\langle [A, B] \rangle|^2/4$.
4. Check that changing the order of rotation $U(R)$ and translation $U(\mathbf{a})$ works as expected:
$$U(R)U_{\mathbf{a}} = U_{R\mathbf{a}}U(R),$$
   where $U_{\mathbf{a}}$ is as in problem 2 and you may restrict to spin-0 case, with $U(R(\mathbf{n}, \omega)) = \exp(-i\omega \mathbf{n} \cdot \hat{\mathbf{L}}/\hbar)$. This can be carried out either by applying the operators to an arbitrary test function $\psi(\mathbf{r})$ or by directly commuting $\hat{\mathbf{p}}$ and $\hat{\mathbf{L}}$ past each other, solve the problem both ways. (The former is very easy, in the latter method it is enough to study infinitesimal $\mathbf{a}$ and ω, only generalize to finite parameters if you find it interesting enough.)
5. Compute the components of the spin operator $\mathbf{S}$ for a spin-1 particle in the basis of S_z eigenstates (*i.e.* write down the matrices for S_x, S_y and S_z.)