KVANTTIMEKANIIKKA I Autumn 2011 Exercise 1

Due at 8.15 on Thursday 15 September 2011

1. Consider the wave function of a particle on the x axis, given by $(a, b > 0)$

$$\psi(x) = \begin{cases} c(x+a)/a & -a \le x \le 0 \\ c(b-x)/b & 0 \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

   (a) Normalize ψ, *i.e.*, find c in terms of a and b such that $\int_{-\infty}^{\infty} \mathrm{d}x\, |\psi(x)|^2 = 1$.

   (b) What is the expectation value of the position and what is the most probable position?

   (c) What is the probability of the particle being on the positive side of the origin?

   Check your result (for questions b and c) in the special case $a = b$.

2. Suppose somebody asked you to find the eigenstates and -values of an "observable" presented in a one-dimensional world by the operator $\hat{A} = \hat{x} + ic\hat{p}$, where c is a real coefficient to make the quantity dimensionally meaningful (*i.e.* change of units). You can see immediately that there is at least a solution for the eigenvalue $a = 0$. Solve the eigenequation for this value. Can this operator correspond to an observable, *i.e.* is it Hermitian?

3. Consider the probability $P_{ab}(t)$ that a particle, whose motion is described by the wave function $\psi(x,t)$, is found somewhere in the interval $a < x < b$ at time t. Show that it obeys the equation

$$\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} = J(a,t) - J(b,t),$$

   where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right).$$

   Investigate the dimension of $J(x,t)$ and think of an interpretation.

4. The wave function of a state is $\psi(x) = N\phi(x)\exp(ip_0x/\hbar)$, where $\phi(x)$ is a quadratically integrable real valued function. Show that the expectation value of the momentum is p_0.

5. Calculate $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$ and $\langle\hat{p}^2\rangle$ for the wave function $\Psi(x) = N\exp(-\alpha|x|)$, where $\alpha > 0$. Be careful if trying to take derivatives of the absolute value $|x|$.