GENERAL RELATIVITY Homework 5 Spring 2012

Due on Monday, February 27th, by 14:15 o'clock.

1. **Transforming basis dual vectors.** The transformation rule for coordinate basis dual vectors
$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x^{\nu'}}\mathrm{d}x^{\nu'}$$
means that you can obtain the transformation treating the dual vectors just as if they were ordinary differentials of coordinates. This may also help in transforming higher-order tensors. Consider the $(0,2)$ tensor (in 2 dimensions)
$$S = \mathrm{d}x \otimes \mathrm{d}x + x\mathrm{d}y \otimes \mathrm{d}y\,.$$
It has the components
$$S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}.$$
Find the components of S in the new coordinates
$$x' = xy\,, \qquad y' = \ln y\,, \qquad (x, y > 0)$$
by inverting this coordinate transformation and writing out $\mathrm{d}x$ and $\mathrm{d}y$ in the new coordinates.

2. **Quotient theorem.** To show that an object is a tensor, it is sufficient to demonstrate that its contraction with arbitrary vectors (or dual vectors) is a tensor. Prove this for the case of a $(1,2)$ tensor, i.e., that if we are given a set of N^3 components $T^\alpha{}_{\beta\gamma}$ such that the contraction $T^\alpha{}_{\beta\gamma}v^\gamma$ transforms as a $(1,1)$ tensor for all vectors v^γ, then the $T^\alpha{}_{\beta\gamma}$ must transform as a $(1,2)$ tensor.

3. **Locally inertial coordinates at the north pole.** The line element of the geometry of a sphere is
$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,,$$
where a is a constant, in the usual spherical coordinates (θ, ϕ). Consider the coordinate transformation
$$x = a\theta\cos\phi\,, \qquad y = a\theta\sin\phi\,,$$
into new coordinates x and y, and show that these are locally inertial coordinates at the north pole, $\theta = 0$. (In the neighborhood of the north pole, x and y are small, so you can write the metric components as an expansion in powers of x and y and include only the lowest powers, if this helps.)

4. **Transformation rule for connection coefficients.** The *covariant derivative* of a vector field v^ν is a $(1,1)$ tensor field, whose components are
$$v^\nu{}_{;\mu} \equiv v^\nu{}_{,\mu} + \Gamma^\nu_{\mu\lambda}v^\lambda\,, \tag{1}$$
where the $\Gamma^\nu_{\mu\lambda}$ are objects called *connection coefficients* (they do not form a tensor). The $\Gamma^\nu_{\mu\lambda}$ are related to the coordinates and the curvature of spacetime, but are independent of the vector field v^ν. Derive the coordinate transformation rule
$$\Gamma^{\alpha'}_{\beta'\gamma'} = \Gamma^\mu_{\nu\rho}X^{\alpha'}_\mu X^\nu_{\beta'}X^\rho_{\gamma'} - X^\nu_{\beta'}X^\rho_{\gamma'}X^{\alpha'}_{\nu\rho}\,,$$
where
$$X^{\alpha'}_{\nu\rho} \equiv \frac{\partial^2 x^{\alpha'}}{\partial x^\nu \partial x^\rho}\,.$$
from the condition that Eq. (1) transforms as a $(1,1)$ tensor.