- Total energy and momentum

The total energy of a system is $\quad E = \int T^{00}\, dV$

The total momentum of a system is $\quad p^i = \int T^{i0}\, dV$

They depend on the frame they are evaluated in.

Together they form a 4-vector (in special relativity only)

- The energy-momentum tensor satisfies the energy-momentum continuity equation

$$\partial_\mu T^{\mu\nu} = 0 \qquad \text{or} \qquad \boxed{T^{\mu\nu}{}_{,\nu} = 0} \tag{1}$$

For a closed system this leads to conservation of total energy

$$\frac{dE}{dt} = \frac{d}{dt}\int T^{00}\, dV = \int \frac{\partial T^{00}}{\partial t}\, dV = \int T^{00}{}_{,0}\, dV$$

$$= -\int T^{0i}{}_{,i}\, dV = -\oint T^{0i}\, dS^i = 0$$

where the surface integral vanishes for a closed system, since there is no energy flux through its boundary.

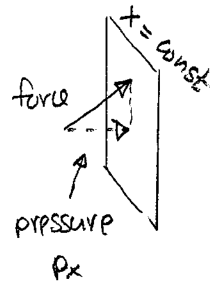
Likewise we get conservation of total momentum

$$\frac{dp^i}{dt} = 0 .$$

- In the fluid rest frame:

$T^{00}$ = proper (rest frame) energy density $\equiv \rho$

$T^{ij}$ describes stress = forces between fluid elements




diagonal terms $\equiv$ pressure $\quad p_1 = T^{11}$

(in the three directions) $\quad p_2 = T^{22}$

$p_3 = T^{33}$

off-diagonal terms $\equiv$ shear

- Dust: particles at rest wrt each other, no forces between them

$\Rightarrow$ all have same 4-velocity $u^\mu$

Number-flux 4-vector $\quad n^\mu \equiv n u^\mu = n(\gamma, \gamma\vec{v})$

(4-hiukkasvirtatiheys) $\quad$ ↑ number density in rest frame

$\Rightarrow \quad n^0 = n\gamma$ = number density in given frame

(since volume element Lorentz-contracted by $\frac{1}{\gamma}$)

$n^i = n\gamma\vec{v} = n^0\vec{v}$ flux of particles in direction $i$

Suppose all particles have same mass $m$

$\Rightarrow$ in rest frame, energy density is $\rho \equiv mn$

in some other frame, energy density is $E n^0 = m\gamma \cdot n\gamma = p^0 \cdot n^0$

both 0-components of a 4-vector

= 0-component of $\underline{p} \otimes \underline{n}$

Define energy-momentum tensor for dust:

$$T^{\mu\nu}_{dust} = p^\mu n^\nu = mnu^\mu u^\nu = \underset{\text{proper energy density}}{\rho}\, u^\mu u^\nu$$

In rest frame $T^{0i}_{dust} = T^{ij}_{dust} = 0$; dust has no pressure, no shear.

- Perfect fluid: frictionless, i.e. cannot support any shear

$\Rightarrow$ pressure is isotopic, $p_1 = p_2 = p_3$

$\Rightarrow$ in rest frame $\quad T^{\mu\nu} = \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}$

General expression for arbitrary inertial frame?

Available: two scalars $\rho, p$ (proper energy density, pressure)

fluid 4-velocity $u^\mu$

spacetime metric $\eta^{\mu\nu}$

$\Rightarrow$ can construct two independent (2,0) tensors: $u^\mu u^\nu$ and $\eta^{\mu\nu}$

$\Rightarrow \quad T^{\mu\nu} = A u^\mu u^\nu + B\eta^{\mu\nu}$

$\Rightarrow$ in rest frame $u^\mu = (1,0,0,0)$ $\quad T^{\mu\nu} = \begin{bmatrix} A-B & & & \\ & B & & \\ & & B & \\ & & & B \end{bmatrix}$

$\Rightarrow \left.\begin{aligned} A-B &= \rho \\ B &= p \end{aligned}\right\} \Rightarrow A = \rho + p$

$$\therefore \boxed{T^{\mu\nu} = (\rho+p)u^\mu u^\nu + p\eta^{\mu\nu}} \qquad (2)$$

- The nature of the perfect fluid is determined by its equation of state $p = p(\rho)$

dust $\quad p = 0 \quad$ (particles whose thermal velocities $v \ll 1$)

radiation $\quad p = \frac{1}{3}\rho \quad$ (massless particles, e.g. isotropic photon gas)

- Apply (1) $T^{\nu\mu}{}_{,\mu} = T^{\mu\nu}{}_{,\mu} = 0$ to a perfect fluid.

- Note first $u^\nu u_\nu = -1 \Rightarrow 0 = (u^\nu u_\nu)_{,\mu} = u^\nu{}_{,\mu} u_\nu + \underbrace{u^\nu u_{\nu,\mu}}_{\eta^{\nu\beta} u_\beta u_{\nu,\mu}} = 2 u^\nu{}_{,\mu} u_\nu$

$$\Rightarrow \quad \underline{u^\nu{}_{,\mu} u_\nu = 0} \qquad (3)$$

whereas $\underline{u^\nu{}_{,\mu} u^\mu} = \dfrac{\partial u^\nu}{\partial x^\mu} \dfrac{dx^\mu}{d\tau} = \dfrac{du^\nu}{d\tau} = \underline{a^\nu}$ is the 4-acceleration

- From (2) $T^{\mu\nu} = \rho u^\mu u^\nu + p u^\mu u^\nu + p \eta^{\mu\nu}$

$$T^{\mu\nu}{}_{,\mu} = (\rho u^\mu)_{,\mu} u^\nu + \rho u^\mu u^\nu{}_{,\mu} + p_{,\mu} u^\mu u^\nu + p u^\mu{}_{,\mu} u^\nu + p u^\mu u^\nu{}_{,\mu} + p_{,\mu} \eta^{\mu\nu} = 0 \qquad (4)$$

by Eq. (1).

- Contract with $u_\nu$ to get a scalar equation

$$u_\nu T^{\mu\nu}{}_{,\mu} = -(\rho u^\mu)_{,\mu} + 0 - \cancel{p_{,\mu} u^\mu} - p u^\mu{}_{,\mu} + 0 + \cancel{p_{,\mu} u^\mu} = 0$$

$$\Rightarrow \quad \underline{(\rho u^\mu)_{,\mu} + p u^\mu{}_{,\mu} = 0} \qquad (5) \qquad \text{Energy continuity}$$

- Subtract $u^\nu \cdot (5)$ from (4):

$$\rho u^\mu u^\nu{}_{,\mu} + p_{,\mu} u^\mu u^\nu + p u^\mu u^\nu{}_{,\mu} + p_{,\mu} \eta^{\mu\nu} = 0$$

$$\Rightarrow \quad \underline{(\rho + p) \underbrace{u^\mu u^\nu{}_{,\mu}}_{a^\nu} + \underbrace{(\eta^{\mu\nu} + u^\mu u^\nu)}_{\text{projection to component orthogonal to } u^\mu} p_{,\mu} = 0} \qquad (6)$$

Pressure gradient accelerates fluid

- Eqs. (5) and (6) are the eqs. of special relativistic hydrodynamics (for a perfect fluid).