Due on Tuesday, January 15th, 2008, by 14:00 o'clock.

1. **Thomson scattering of circular polarization.** The differential cross section of Thomson scattering is
$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi}\left|\hat{\epsilon}' \cdot \hat{\epsilon}^*\right|^2$$
where $\hat{\epsilon}'$ and $\hat{\epsilon}$ are the polarization vectors of the incoming and outgoing photons. In general they are complex unit vectors, and can be expressed, e.g., in the scattering plane basis $\hat{\epsilon}_{\|}$, $\hat{\epsilon}_{\perp}$ as $a_1 e^{i\alpha_1}\hat{\epsilon}_{\|} + a_2 e^{i\alpha_2}\hat{\epsilon}_{\perp}$. Define the Stokes parameters with respect to the scattering plane basis, and find out
   (a) how Stokes parameters $I(\vec{q}')$, $V(\vec{q}')$ scatter into $I(\vec{q})$, $V(\vec{q})$
   (b) how Stokes parameters $I(\vec{q}')$, $Q(\vec{q}')$ scatter into $I(\vec{q})$, $V(\vec{q})$.

2. **Scattering matrix** The scattering matrix for the Stokes parameter "vector" $(I_{\|}, I_{\perp}, U)$ is
$$S(\beta) = \frac{3}{2}\begin{bmatrix} \cos^2\beta & & \\ & 1 & \\ & & \cos\beta \end{bmatrix}$$
Convert it to the scattering matrix for $\bar{T} \equiv (I, Q+iU, Q-iU)$ (Q and U defined with respect to the scattering plane).

3. **Scattering of anisotropic unpolarized radiation.** Consider Thomson scattering of unpolarized radiation. Express the gain terms $dI(\vec{q})/dt$, $dQ(\vec{q})/dt$, $dU(\vec{q})/dt$ in terms of the multipoles of the directional dependence of the incoming radiation $I(q, \hat{n}')$ in the coordinate system where $\hat{z} \upuparrows \vec{q}$. Show that the outgoing radiation is isotropic and unpolarized if the quadrupole of the incoming radiation vanishes.

4. **Symmetries of Wigner D-functions.** Derive the symmetry relation
$$D^{\ell}_{mm'}(\alpha, \beta, \gamma) = D^{\ell *}_{-m,-m'}(\alpha, -\beta, \gamma)\,,$$
Eq. (11) of lecture notes Chapter C4, from the relations given before that in the lecture notes.

5. **Spin 2 spherical harmonics.** Derive the explicit forms of
$${}_2Y_2^m(\theta, \phi) \equiv \frac{N_2}{\sqrt{2}}\left(W_{2m} + iX_{2m}\right)\,,$$
from the equations for tensor spherical harmonics in Chapter Y3 and the formulae for the ordinary Y_ℓ^m given in Chapter F6. Verify that they agree with
$${}_sY_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, e^{im\phi} d^{\ell}_{m,-s}(\theta)$$
and the formulae for $d^2_{mm'}(\theta)$ in Table 4.6 of Varshalovich et al. (reproduced on page C4.3 of lecture notes).

6. **Free streaming of polarization**. Find the polarization multipoles $E_\ell^{m\prime}$ and $B_\ell^{m\prime}$ of the free streaming equation
$$\frac{\partial \bar{T}(\eta, \hat{n})}{\partial \eta} = -ikn_3\bar{T}(\eta, \hat{n})\,.$$
(Consider separately the case of $\ell = 2$). Explain, using this result, why vector and tensor perturbations produce both E-mode and B-mode polarization, but scalar perturbations produce only E-mode polarization.